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LETTER TO THE EDITOR

On the BPS limit in the classical SU(2) gauge theory

Y Yang

Department of Mathematics, University of New Mexico, Albuquerque, NM 87131, USA

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Abstract. It is noted that as the Higgs self-coupling parameter $\lambda \to 0$ the classical 't Hooft-Polyakov SU(2) monopoles approach the Bogomol'nyi-Prasad-Sommerfield solution uniformly over entire space.

The 't Hooft-Polyakov monopoles of the classical SU(2) Yang-Mills-Higgs (YMH) theory with the Higgs field in the adjoint or triplet representation of the gauge group are in the form [1, 2]

$$\phi^a(x) = \frac{x^a}{r} h_\lambda(r)$$

$$A_0^a(x) = 0 \qquad A_i^a(x) = \varepsilon_{aij} \frac{x^j}{r^2}(1 - k_\lambda(r)) \qquad a, i, j = 1, 2, 3$$

where $r = |x|$ and (h_λ, k_λ) is a solution of the reduced YMH equations

$$\begin{cases} (r^2 h')' = 2k^2 h + \lambda r^2 (h^2 - 1) h \\ k'' = h^2 k + \dfrac{k^2 - 1}{r^2} k \end{cases} \qquad r > 0 \qquad (1_\lambda)$$

which minimises, among all pairs of such real scalar fields h, k that satisfy the boundary condition

$$\lim_{r\to 0} h(r) = 0 \qquad \lim_{r\to\infty} h(r) = 1 \qquad \lim_{r\to 0} k(r) = 1 \qquad \lim_{r\to\infty} k(r) = 0 \qquad (2)$$

the normalised YMH energy

$$E_\lambda(h, k) = \frac{1}{2}\int_0^\infty \mathrm{d}r \left(\frac{1}{2} r^2 h'^2 + k'^2 + h^2 k^2 + \frac{1}{2r^2}(k^2 - 1)^2 + \frac{\lambda}{4} r^2 (h^2 - 1)^2\right). \qquad (3)$$

The existence of such solutions was first established by Tyupkin *et al* [3] and the regularity verified in the work of Rawnsley [4]. In the Bogomol'nyi-Prasad-Sommerfield (BPS) non-physical limit $\lambda = 0$ corresponding to vanishing Higgs potential and zero Higgs mass, an explicit solution of (1_0) was found by guesswork [5, 6]:

$$h_0(r) = \coth r - \frac{1}{r} \qquad k_0 = \frac{r}{\sinh r} \qquad r > 0.$$

The pair (h_0, k_0) gives rise to the well known BPS monopole of unit magnetic charge. On the other hand, if $\lambda > 0$, none has succeeded in obtaining an explicit solution of (1_λ) and (2). The purpose of the present letter is to note that, in fact,

$$\sup_{0<r<\infty} \{|h_\lambda(r) - h_0(r)| + |k_\lambda(r) - k_0(r)|\} \to 0 \qquad \text{as } \lambda \to 0. \tag{4}$$

This result, which is suggested by the numerical study of Bais and Primack [7] and partially justified in the paper of Chernavskii and Kerner [8], may be useful in determining the global behaviour of the 't Hooft-Polyakov monopoles for small values of λ.

In order to get the convergence control (4), we need to derive some sufficient λ-independent estimates of the solutions (h_λ, k_λ), for $0 < \lambda < 1$, say. For simplicity, in what follows, C will denote such a generic positive constant independent of λ that may vary its value at different places.

The first two lemmas in the following indicate that the boundary behaviour of h_λ, k_λ at $r = \infty$ is uniform with respect to the parameter λ.

Lemma 1. There holds the λ-independent estimate

$$|h_\lambda(r) - 1| \leq C r^{-1/2} \qquad r > 0, 0 < \lambda < 1. \tag{5}$$

Proof. By taking a suitable trial function pair, it is easily seen that

$$E_\lambda(h_\lambda, k_\lambda) \leq C \qquad 0 < \lambda < 1 \tag{6}$$

for some C. Thus (5) follows from the simple inequality

$$\begin{aligned} |h_\lambda(r) - 1| &\leq \left| \int_r^\infty h'_\lambda(\rho)\, d\rho \right| \\ &\leq \left(\int_r^\infty \rho^{-2}\, d\rho \right)^{1/2} \left(\int_0^\infty \rho^2 h'^2_\lambda\, d\rho \right)^{1/2} \end{aligned} \tag{7}$$

(3) and (6).

Lemma 2. There exists an $r_0 > 0$, independent of $0 < \lambda < 1$, so that

$$k_\lambda^2(r) \leq e^{r_0 - r} \qquad r \geq r_0. \tag{8}$$

Proof. From (1_λ) we have

$$(k_\lambda^2)'' \geq 2\left(h_\lambda^2 + \frac{1}{r^2}(k_\lambda^2 - 1) \right) k_\lambda^2. \tag{9}$$

It can be seen that $k_\lambda^2 \leq 1$. Otherwise, if there were $r > 0$ so that $k_\lambda^2(r) > 1$, then the behaviour (2) would imply that there were $0 \leq a < b$ to make $k_\lambda^2(a) = k_\lambda^2(b) = 1$ and $k_\lambda^2(r) > 1$, $r \in (a, b)$. Let $r' \in (a, b)$ satisfy $k_\lambda^2(r') = \max_{r \in (a,b)} k_\lambda^2(r)$. By virtue of (9), one would get $k_\lambda^2(r') = 0$, a contradiction.

Thus, using lemma 1, one can find an $r_0 > 0$ independent of λ such that $(k_\lambda^2)'' \geq k_\lambda^2$, $r > r_0$. Consequently $k_\lambda^2(r) \leq e^{r_0 - r}$ for $r \geq r_0$. The lemma is proved.

The next two lemmas imply that the boundary behaviour of h_λ, k_λ as $r \to 0$ is also uniform.

Lemma 3. For $0<\lambda<1$, there is a constant C, so that

$$|h_\lambda(r)|\leqslant Cr^{1/2} \qquad 0<r<1.$$

Proof. The argument is a specialisation of that given in Rawnsley [4].

Set $f_\lambda(r)=r^{-1}h_\lambda(r)$. Due to (1_λ), f_λ satisfies the equation

$$(r^4f'_\lambda)'=2r^2(k_\lambda^2-1)f_\lambda+\lambda r^3(h_\lambda^2-1)h_\lambda.$$

Since $r^4f'_\lambda(r)=r^3h'_\lambda(r)-r^2h_\lambda(r)\to 0$ as $r\to 0$, we have

$$r^4f'_\lambda(r)=\int_0^r \mathrm{d}\rho\left[\left(\frac{2}{\rho}(k_\lambda^2-1)+\lambda\rho(h_\lambda^2-1)\right)\rho^3 f_\lambda\right].$$

By virtue of the above, the Schwarz inequality, and (6), we obtain

$$|r^4f'_\lambda(r)|\leqslant C\left(\int_0^r \rho^6 f_\lambda^2(\rho)\,\mathrm{d}\rho\right)^{1/2}. \tag{10}$$

On the other hand, (7) gives us the estimate $|h_\lambda(r)|\leqslant Cr^{-1/2}$ or $|f_\lambda(r)|\leqslant Cr^{-3/2}$. So (10) implies $|f'_\lambda(r)|\leqslant Cr^{-2}$, $0<r<1$, and, in particular, $|f_\lambda(r)|\leqslant Cr^{-1}$, $0<r<1$. Applying this latter bound in (10) again we conclude with $|f'_\lambda(r)|\leqslant Cr^{-3/2}$, which in turn implies $|f_\lambda(r)|\leqslant Cr^{-1/2}$, $0<r<1$, and the desired estimate follows.

Lemma 4. There is a C so that $|k_\lambda(r)-1|\leqslant Cr^{1/2}$, $0<r<1$.

Proof. One may use an argument similar to that for lemma 1.

The uniqueness result stated below ensures that, as $\lambda\to 0$, $(h_\lambda, k_\lambda)\to(h_0, k_0)$.

Lemma 5. For $\lambda=0$, (h_0, k_0) is the only solution of (1_0) satisfying the boundary condition (2).

This uniqueness result was first observed in the numerical work of Frampton [9] and later proved by Maison [10] where the main argument is to show that a finite energy solution of (1_0) and (2) is a solution of the Bogomol'nyi equations

$$\begin{cases} h'+\dfrac{1}{r^2}(k^2-1)=0 \\ k'+hk=0 \end{cases} \qquad r>0 \tag{11}$$

derived from the duality condition $B_i^a+D_i\phi^a=0$ with $B_i^a=-\frac{1}{2}\varepsilon_{ikl}F_{kl}^a$. The fact that (h_0, k_0) is the only solution of (11) subject to (2) may be well-accepted. However, since we were unable to locate a proof of this fact, here we choose to furnish one for completeness.

Let (h, k) be a solution of (11) and (2). First the property $k^2\leqslant 1$ tells us that $h(r)$ is a non-decreasing function. Hence $h(r)\geqslant 0$, $r\geqslant 0$. The second equation in (11) then indicates that $k(r)>0$, $r\geqslant 0$. Therefore we are allowed to introduce the transform $u=\ln k$ and reduce (11) into

$$u''=\frac{1}{r^2}(\mathrm{e}^{2u}-1) \qquad r>0. \tag{12}$$

Since $u_0 = \ln k_0$ also satisfies (12), we obtain

$$(u-u_0)'' = \frac{1}{r^2}(e^{2u} - e^{2u_0})$$

and, consequently, for $0 < r_1 < r_2 < \infty$,

$$0 \leqslant \int_{r_1}^{r_2} dr\left(\frac{1}{r^2}(e^{2u} - e^{2u_0})(u-u_0)\right) = M(r)\Big|_{r_1}^{r_2} - \int_{r_1}^{r_2} (u'-u_0')^2\, dr \qquad (13)$$

where

$$M(r) = (u-u_0)(u'-u_0')(r).$$

We claim that

$$\lim_{r\to 0} M(r) = \lim_{r\to\infty} M(r) = 0. \qquad (14)$$

The first limit in (14) is easily seen from the expression

$$M(r) = -[\ln k(r) - \ln k_0(r)][h(r) - h_0(r)].$$

To obtain a verification of the second limit, we proceed as follows.

We may assume $w'(r) \neq 0$ for $r > 0$ where $w = u - u_0$. Otherwise, if there is a point $r_0 > 0$ so that $w'(r_0) = 0$, then w satisfies

$$w'' = \frac{1}{r^2}(e^{2u} - e^{2u_0}) = \frac{1}{r^2}\xi(u, u_0)w \qquad r > 0$$

$$w(0) = w'(r_0) = 0$$

with $\xi > 0$. Therefore w cannot have an absolute extremum at $r = r_0$. The maximum principle tells us that $w(r) = 0$ for $r \in (0, r_0)$. Thus $w \equiv 0$ for $r > 0$. In particular (14) is true.

Hence, we now assume $w'(r) \neq 0$ which implies that $w(r)$ is monotone. So $\lim_{r\to\infty} w(r)$ exists or equals $\pm\infty$. The former case still verifies (14). Thus it remains to argue for the latter case.

We have, using L'Hospital's principle, (11) and (2),

$$\lim_{r\to\infty} \frac{w(r)}{r} = \lim_{r\to\infty} w'(r) = -\lim_{r\to\infty}(h(r) - h_0(r)) = 0$$

and

$$\lim_{r\to\infty} rw'(r) = \lim_{r\to\infty} \frac{w''}{-r^{-2}} = -\lim_{r\to\infty}(k^2(r) - k_0^2(r)) = 0.$$

Consequently, we get $\lim_{r\to\infty} w(r)w'(r) = 0$. This proves the second limit in (14).

Now, letting $r_1 \to 0$, $r_2 \to \infty$ in (13) we find $u \equiv u_0$, namely, $(h, k) = (h_0, k_0)$. Lemma 5 follows.

Finally, from (6), lemmas 1-4 and the structure of E_λ, it is easily seen that $\{(h_\lambda, k_\lambda)\}_{0<\lambda<1}$ is a precompact subset of $C^0[0, \infty)$. Lemma 5 and a simple argument then prove (4), i.e. $(h_\lambda, k_\lambda) \to (h_0, k_0)$ in $C^0[0, \infty)$ as $\lambda \to 0$.

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